

# Distinction of regular depth-zero supercuspidal $L$ -packets

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## Abstract

In this paper, we study the relative Langlands program formulated by Dipendra Prasad for Galois symmetric spaces. Under certain assumptions, we confirm the necessary conditions in Prasad's conjecture for regular depth-zero supercuspidal  $L$ -packets in the sense of DeBacker-Reeder and Kaletha.

## 1 Introduction

**Prasad's conjecture.** Let  $F$  be a nonarchimedean local field of characteristic 0. Let  $G$  be a connected reductive group defined over  $F$ , and  $H$  a spherical subgroup of  $G$ . We say that an irreducible admissible representation  $\pi$  of  $G(F)$  is  $H(F)$ -distinguished if the space  $\text{Hom}_{H(F)}(\pi, 1)$  is nonzero. More generally, for a character  $\omega$  of  $H(F)$ , we say that  $\pi$  is  $\omega$ -distinguished if  $\text{Hom}_{H(F)}(\pi, \omega)$  is nonzero. Under some assumptions on  $G$  and the spherical variety  $X = G/H$ , Sakellaridis and Venkatesh [SV] formulate a relative Langlands program whose aim is to understand the spectral decomposition of  $L^2(X(F))$ . Roughly speaking, they conjecture that if  $\pi$  is  $H(F)$ -distinguished then  $\pi$  should be a Langlands functorial lift from the dual group  $G_X$  attached to  $X$ . When  $X$  is a Galois symmetric space, Prasad [Pra] makes a more precise conjecture whose goal is to give sufficient and necessary conditions, and even multiplicity formulas, for representations to be distinguished.

Now let us introduce the setting of Prasad's conjecture. Let  $E$  be a quadratic field extension of  $F$ ,  $H$  a connected quasi-split reductive group over  $F$ , and  $G = \text{R}_{E/F}H$  the Weil restriction of  $H$  with respect to the extension  $E/F$ . Twisting the group  $H$  by sending the nontrivial element  $\sigma$  of  $\text{Gal}(E/F)$  to the Chevalley involution  $C$ , we get a quasi-split reductive group  $H^{\text{op}}$  over  $F$ . In other words, the group  $H^{\text{op}}$  is isomorphic to  $H$  over  $E$ , and its group of  $F$ -rational points is

$$H^{\text{op}}(F) = \{g \in H(E) : C(g) = \sigma(g)\}.$$

It is the group  $H^{\text{op}}$  that serves the dual group  $G_X$  in this framework. Let  $W_F$  be the Weil group of  $F$ ,  $\widehat{G}$  the complex Langlands dual group for  $G$ , and  ${}^L G = \widehat{G} \rtimes W_F$  the corresponding  $L$ -group. The  $L$ -group  ${}^L H^{\text{op}}$  of  $H^{\text{op}}$  admits an  $L$ -embedding  ${}^L H^{\text{op}} \rightarrow {}^L G$ . We refer the reader to §3.3 for more information about  $H^{\text{op}}$ .

Let  $H_\alpha$  be a pure inner twist of  $H$  over  $F$  such that  $\text{R}_{E/F}H_\alpha = G$ . Then the  $F$ -structure of  $H_\alpha$  gives rise to an action of  $\sigma \in \text{Gal}(E/F)$  on  $G$  over  $F$ .

We denote this action by  $\sigma_\alpha$ , and if  $H_\alpha = H$  we simply write  $\sigma$  instead of  $\sigma_\alpha$ . Note that, identifying  $H_\alpha(E) = H(E) = G(F)$ , there exists  $g_\alpha \in G(F)$  such that  $\sigma_\alpha = \text{Ad}(g_\alpha) \circ \sigma$ . For any reductive group  $A$  over  $F$ , Prasad considers a character  $\omega_A$  of  $A(F)$  associated with the quadratic extension  $E$ . The character  $\omega_A$  is trivial or quadratic. Let  $\pi$  be an irreducible admissible representation of  $G(F)$ . Prasad gives a conjectural criterion of  $\omega_{H_\alpha}$ -distinction for  $\pi$  in terms of the Langlands parameter of  $\pi$ . Suppose that the conjectural local Langlands correspondence holds for  $G$ . Then  $\pi$  lies in an  $L$ -packet  $\Pi_\varphi(G)$ , where  $\varphi : W_F \rightarrow {}^L G$  is a Langlands parameter and  $\Pi_\varphi(G)$  is a finite set of equivalence classes of irreducible representations of  $G(F)$  corresponding to the parameter  $\varphi$ . Let  $\pi^{\sigma_\alpha}$  be the Galois conjugate of  $\pi$  with respect to  $\sigma_\alpha$ . Since  $\sigma_\alpha$  and  $\sigma$  differ by conjugation in  $G(F)$ , we have  $\pi^{\sigma_\alpha} \simeq \pi^\sigma$ . Therefore the equivalence class of  $\pi^{\sigma_\alpha}$  does not depend on  $\alpha$ . We denote by  $\Pi_\varphi^\sigma(G)$  the  $L$ -packet that  $\pi^\sigma$  belongs to. On the other hand, we denote by  $\pi^\vee$  the contragredient of  $\pi$ , and denote by  $\Pi_\varphi^\vee(G)$  the  $L$ -packet where  $\pi^\vee$  lies in. As indicated by the notation, we will explain in §§2.3–2.4 that both  $\Pi_\varphi^\vee(G)$  and  $\Pi_\varphi^\sigma(G)$  do not depend on  $\pi$ . In other words, we have

$$\Pi_\varphi^\vee(G) = \{\tau^\vee : \tau \in \Pi_\varphi(G)\} \quad \text{and} \quad \Pi_\varphi^\sigma(G) = \{\tau^\sigma : \tau \in \Pi_\varphi(G)\}.$$

Part of [Pra, Conjecture 2] can be stated as follows.

**Conjecture 1.1.** If  $\pi$  is  $\omega_{H_\alpha}$ -distinguished, we have

1.  $\Pi_\varphi^\vee(G) = \Pi_\varphi^\sigma(G)$ .
2. The parameter  $\varphi$  factors through  ${}^L H^{\text{op}}$ .

Let  $Z_\varphi$  be the centralizer of  $\varphi$  in  $\widehat{G}$ ,  $C_\varphi$  the group of connected components of  $Z_\varphi$ , and  $\mu$  an irreducible representation of  $C_\varphi$  corresponding to  $\pi$ . Prasad [Pra, Conjecture 2] also provides a sufficient condition for  $\omega_{H_\alpha}$ -distinction in terms of  $\mu$ , and a conjectural formula for the multiplicity

$$\dim \text{Hom}_{H_\alpha}(\pi, \omega_{H_\alpha})$$

in terms of fibers of functorial lifts. Moreover, besides the nonarchimedean case, the archimedean case  $F = \mathbb{R}$  is equally considered in [Pra]. See [Pra, Conjecture 2] for the precise statements. Previous works by other authors for specific  $H$  or  $\pi$  are well discussed in [Pra].

**Main results.** Assume that the characteristic of the residue field of  $F$  is not 2. In this paper, we study Conjecture 1.1 in the case where  $H$  is unramified,  $E$  unramified over  $F$ , and  $\pi$  in a regular depth-zero supercuspidal  $L$ -packet. Regular depth-zero supercuspidal  $L$ -packets are first constructed by DeBacker and Reeder [DR09] for pure inner twists of unramified  $p$ -adic groups. Later Kaletha [Kal14] reinterprets and enlarges their construction for extended pure inner twists. These two kinds of  $L$ -packets are the same on pure inner twists, and both satisfy the refined local Langlands conjectures (cf. [Kal, Conjectures E and F]). We will adopt the formalism of [Kal14], and consider all the extended pure inner twists  $H_a$  of  $H$  over  $F$ . This formulation enables us to treat the problem of distinction not only for quasi-split groups but also their extended pure inner twists.

Now let  $H$  be unramified. We take an extended pure inner twist  $H_a$  of  $H$ . Let  $G_a = R_{E/F}H_a$  and  $\pi$  be an irreducible admissible representation of  $G_a(F)$ . We always suppose that  $\pi$  is in a regular depth-zero supercuspidal  $L$ -packet. The following theorem (Theorem 1.2) is about the representation  $\pi$  itself rather than its  $L$ -packet.

**Theorem 1.2.** *If  $\pi$  is  $\omega_{H_a}$ -distinguished, we have*

$$\pi^\vee \simeq \pi^\sigma.$$

We mainly use Hakim-Murnaghan theory to prove Theorem 1.2, and the proof heavily relies on the construction of  $\pi$ . Now let  $\varphi : W_F \rightarrow {}^L G$  be a Langlands parameter attached to  $\pi$ . To prove Conjecture 1.1, our strategy is to deduce the properties of  $\varphi$  from Theorem 1.2. The regular depth-zero supercuspidal  $L$ -packet  $\Pi_\varphi$  associated with  $\varphi$  is defined to be the disjoint union

$$\bigsqcup \Pi_\varphi(G_b)$$

where  $G_b$  runs over extended pure inner twists of  $G$  over  $F$ , and  $\Pi_\varphi(G_b)$  is the  $L$ -packet of  $G_b(F)$  attached to  $\varphi$ . Each  $L$ -packet  $\Pi_\varphi(G_b)$  is explicitly constructed. We will sometimes call  $\Pi_\varphi$  an *enlarged  $L$ -packet* to distinguish it from  $L$ -packets  $\Pi_\varphi(G_b)$ . From the definition of  $\Pi_\varphi$  we see that the word “enlarged” means that the  $L$ -packets of extended pure inner twists are grouped together and thus can be studied uniformly. Let  $Z_\varphi$  be the centralizer of  $\varphi$  in  $\widehat{G}$ , which is an abelian group in our situation. There is a nice parametrization of  $\Pi_\varphi$  by the group  $Z_\varphi^D$  of characters of  $Z_\varphi$ . After fixing a Whittaker datum, the correspondence

$$\iota : Z_\varphi^D \longrightarrow \Pi_\varphi$$

is canonical. We call  $\iota$  an *enlarged local Langlands correspondence*. For  $\mu \in Z_\varphi^D$  such that  $\iota(\mu) = \pi$  we call  $(\varphi, \mu)$  a *refined Langlands parameter* of  $\pi$ . We will study the refined Langlands parameters of  $\pi^\vee$  and  $\pi^\sigma$  (see Proposition 2.3), and obtain a relation between them (see Theorem 3.5) if  $\pi$  is  $\omega_{H_a}$ -distinguished. With this relation at hand, we can answer Conjecture 1.1. We say that an extended pure inner twist  $G_b$  of  $G$  comes from  $H$  if  $G_b = R_{E/F}H_b$  for some extended pure inner twist  $H_b$  of  $H$ . We consider a subset  $\Pi_\varphi^\circ$  of  $\Pi_\varphi$  which is a disjoint union of  $\Pi_\varphi(G_b)$  for those  $G_b$  coming from  $H$ . As before it makes sense to define the Galois conjugate  $\tau^\sigma$  for  $\tau \in \Pi_\varphi^\circ$ . Set

$$\Pi_\varphi^{\circ, \vee} = \{\tau^\vee : \tau \in \Pi_\varphi^\circ\} \quad \text{and} \quad \Pi_\varphi^{\circ, \sigma} = \{\tau^\sigma : \tau \in \Pi_\varphi^\circ\}.$$

The advantage of the enlarged local Langlands correspondence is that it enables us to compare  $\Pi_\varphi^{\circ, \vee}$  with  $\Pi_\varphi^{\circ, \sigma}$ , and not merely  $\Pi_\varphi^\vee(G_a)$  with  $\Pi_\varphi^\sigma(G_a)$ .

**Theorem 1.3.** *If  $\pi$  is  $\omega_{H_a}$ -distinguished, we have*

1.  $\Pi_\varphi^{\circ, \vee} = \Pi_\varphi^{\circ, \sigma}$ .
2. The parameter  $\varphi$  factors through  ${}^L H^{\text{op}}$ .

In fact, the sets  $\Pi_\varphi^{\circ, \vee}$  and  $\Pi_\varphi^{\circ, \sigma}$  are subsets of certain enlarged  $L$ -packets. The  $L$ -group  ${}^L G$  is equipped with two involutions  $C$  and  $\delta$ , which are related to the actions of taking contragredient and Galois conjugate on the representations

respectively. Thus, starting from the parameter  $\varphi$ , we have two other parameters  $C \circ \varphi$  and  $\delta \circ \varphi$ . Kaletha [Kal13, §5] shows that  $\Pi_{\varphi}^{\vee}(G_b) = \Pi_{C \circ \varphi}(G_b)$  for any extended pure inner twist  $G_b$  of  $G$ . Therefore  $\Pi_{\varphi}^{\circ, \vee}$  is a subset of  $\Pi_{C \circ \varphi}$ . On the other hand, we will show that  $\Pi_{\varphi}^{\sigma}(G_b) = \Pi_{\delta \circ \varphi}(G_b)$  for any extended pure inner twist  $G_b$  coming from  $H$  (see Proposition 2.1). Thus  $\Pi_{\varphi}^{\circ, \sigma}$  is a subset of  $\Pi_{\delta \circ \varphi}$ . Hence Theorem 1.4 below (also see Corollary 3.6) implies Theorem 1.3, and offers positive evidence to Conjecture 1.1. The proof of Theorem 1.4 is a direct consequence of Theorem 3.5 and the description of  ${}^L H^{\text{op}}$ .

**Theorem 1.4.** *If  $\pi$  is  $\omega_{H_a}$ -distinguished, we have*

1.  $\Pi_{C \circ \varphi} = \Pi_{\delta \circ \varphi}$ .
2. *The parameter  $\varphi$  factors through  ${}^L H^{\text{op}}$ .*

**Organization of this article.** Conjecture 1.1 only involves representations of quasi-split groups, while our treatment deals with more general reductive groups. We recall the local Langlands conjectures for quasi-split groups in §2.1, and the enlarged local Langlands correspondence for extended pure inner twists in §2.2 where we only focus on regular depth-zero supercuspidal representations. In the subsequent sections §§2.3–2.4, we show the relationship between actions (i.e. contragredient and Galois conjugate) on the representations with actions (i.e. involutions  $C$  and  $\delta$ ) on the parameters. Main theorems of the paper and their proof are given in §3. The proof of Theorem 1.2 relies on Proposition 3.2 whose proof involves Hakim-Murnaghan theory for distinguished tame supercuspidal representations and the theory for distinguished representations over finite fields. Theorems 1.3 and 1.4 follow from Theorem 3.5 whose proof relies on Theorem 1.2 and Proposition 2.3.

**Notation and conventions.** Let  $F$  be a  $p$ -adic field (i.e. a finite extension of  $\mathbb{Q}_p$ ) with ring of integers  $O_F$  and residue field  $k_F$ . For the construction of regular supercuspidal  $L$ -packets, we assume that  $p$  is odd. Let  $E$  be an unramified quadratic field extension of  $F$  with ring of integers  $O_E$  and residue field  $k_E$ . We write  $W_F$  and  $W_E$  for the Weil groups of  $F$  and  $E$  respectively, and write  $I_F$  for the inertia subgroup of  $W_F$ . Denote by  $\sigma$  the nontrivial automorphism in  $\text{Gal}(E/F)$ . Fix an algebraic closure  $\bar{F}$  such that  $E \subset \bar{F}$ , and denote by  $\Gamma$  the absolute Galois group  $\text{Gal}(\bar{F}/F)$ . Let  $F^u$  be the maximal unramified extension of  $F$  in  $\bar{F}$ .

If  $H$  is an algebraic group over  $F$ , we use  $R_{E/F}H$  to denote its Weil restriction attached to the extension  $E/F$ . Thus  $R_{E/F}H$  is an algebraic group over  $F$  whose group of  $F$ -rational points is  $H(E)$ . The  $F$ -rational structure of  $H$  gives rise to an action of  $\text{Gal}(E/F)$  on  $R_{E/F}H$ . Denote by  $\theta$  the involution on  $R_{E/F}H$  induced by  $\sigma$ . If  $W$  is a subgroup of  $R_{E/F}H$ , by abuse of notation, we denote the image of  $W$  under  $\theta$  by  $W^{\sigma}$ , and denote by  $W^{\theta}$  the subgroup of fixed points. If  $(\pi, V_{\pi})$  is a representation of  $H(E)$  where  $V_{\pi}$  is the underlying space of  $\pi$ , the Galois conjugate  $\pi^{\sigma}$  of  $\pi$  is a representation of  $H(E)$  with underlying space  $V_{\pi}$ , defined by  $\pi^{\sigma}(g)v = \pi(g^{\sigma})v$  for  $v \in V_{\pi}$ . We will use similar notation when we discuss objects over finite fields.

For a connected reductive group  $G$  over  $F$ , we denote by  $\mathcal{B}^{\text{red}}(G, F)$  the reduced Bruhat-Tits building of  $G(F)$ . For any  $x \in \mathcal{B}^{\text{red}}(G, F)$ , we write

$G(F)_{x,0}$  for the parahoric subgroup corresponding to  $x$ ,  $G(F)_{x,0+}$  for its pro-unipotent radical, and  $\mathbf{G}$  for the corresponding connected reductive group over  $k_F$ . For any unramified maximal torus  $S$  of  $G$ , we denote the intersection  $\mathcal{A}^{\text{red}}(S, F^u) \cap \mathcal{B}^{\text{red}}(G, F)$  by  $\mathcal{A}^{\text{red}}(S, F)$ , where  $\mathcal{A}^{\text{red}}(S, F^u)$  is the reduced apartment of  $S$  in  $\mathcal{B}^{\text{red}}(G, F^u)$ .

If  $A$  is a group, we use  $A^D$  to denote  $\text{Hom}(A, \mathbb{C}^\times)$ , and use  $\text{Irr}(A)$  to denote the set of equivalence classes of irreducible representations of  $A$ . If  $A$  is a topological group, we use  $\pi_0(A)$  to denote its group of connected components.

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## 2 $L$ -packets

### 2.1 Local Langlands conjectures

There are several versions of the local Langlands conjectures, that is, the version for quasi-split groups, the enlarged version for pure inner twists of quasi-split groups, and the recent version for extended pure inner twists and rigid inner twists of quasi-split groups. We refer to [Kal] for an excellent survey on the relation between these versions of conjectures. In this subsection, we briefly review the local Langlands conjecture for quasi-split groups.

Let  $G$  be a connected reductive group (not necessary be quasi-split) defined over  $F$ ,  $\widehat{G}$  the complex Langlands dual group of  $G$ , and  ${}^L G = \widehat{G} \rtimes W_F$  the Weil-form  $L$ -group of  $G$ . The Langlands parameters (considered in this paper) are continuous homomorphisms

$$\varphi : W_F \rightarrow {}^L G$$

which are sections of the natural projection  ${}^L G \rightarrow W_F$ . For each  $L$ -parameter  $\varphi$  up to  $\widehat{G}$ -conjugate, the basic form of the local Langlands conjecture predicts the existence of a finite set  $\Pi_\varphi(G)$ , called an  $L$ -packet, of equivalence classes of irreducible admissible representations of  $G(F)$ . If two parameters  $\varphi$  and  $\varphi'$  are not  $\widehat{G}$ -conjugate, the packets  $\Pi_\varphi(G)$  and  $\Pi_{\varphi'}(G)$  should be disjoint. Each irreducible admissible representation  $\pi$  of  $G(F)$  should belong to a unique packet.

Now we suppose further that  $G$  is quasi-split. The refined local Langlands conjecture says that we can parameterize representations in  $\Pi_\varphi(G)$  in terms of the information of the parameter  $\varphi$ . To be more precise, let  $Z_\varphi$  be the centralizer of  $\varphi$  in  $\widehat{G}$ , and  $Z(\widehat{G})$  the center of  $\widehat{G}$ . Then there should exist a bijective map

$$\iota : \text{Irr} \left( \pi_0 \left( Z_\varphi / Z(\widehat{G})^\Gamma \right) \right) \longrightarrow \Pi_\varphi(G), \quad (1)$$

and this map is unique in the following sense. Recall that a Whittaker datum for  $G$  is a  $G(F)$ -conjugacy class of pairs  $(B, \psi)$ , where  $B$  is a Borel subgroup of  $G$  defined over  $F$  with unipotent radical  $U$ , and  $\psi$  is a non-degenerate character  $U(F) \rightarrow \mathbb{C}^\times$ . Given a Whittaker datum  $(B, \psi)$ , an admissible representation  $\pi$  is called  $(B, \psi)$ -generic if  $\text{Hom}_{U(F)}(\pi, \psi) \neq 0$ . Fix a Whittaker datum  $(B, \psi)$ . The

map  $\iota$  should be unique in the sense that it sends the trivial representation in  $\text{Irr}\left(\pi_0\left(Z_\varphi/Z(\widehat{G})^\Gamma\right)\right)$  to the conjecturally unique  $(B, \psi)$ -generic representation in  $\Pi_\varphi(G)$ , and satisfies the endoscopic character identities (which we will not review). We denote by  $\iota_{B, \psi}$  this conjectural map.

## 2.2 Regular depth-zero supercuspidal $L$ -packets

**Extended pure inner twists.** To parameterize representations of non-quasi-split groups, one has to consider all inner twists of the quasi-split group  $G$ , whose isomorphism classes are parameterized by  $H^1(F, G_{\text{ad}})$  where  $G_{\text{ad}}$  is the adjoint group of  $G$ . However, inner twists are not rigid enough to ensure the existence of a bijection as (1), since the automorphism group of an inner twist is bigger than its inner automorphism group. Vogan introduces the notion of pure inner twists to rigidify inner twists. The remaining problem is that not all of the inner twists can be rigidified to be pure. To include more inner twists which can be rigidified, based on his work on isocrystals with additional structure, Kottwitz introduces the notion of extended pure inner twists. We adopt the formulation of extended pure inner twists in [Kal14, §2] and refer to [Kot] for a complete introduction.

Let  $L$  be the completion of  $F^u$ , and  $\bar{L}$  a fixed algebraic closure of  $L$  such that  $\bar{F} \subset \bar{L}$ . There exists a subset  $Z^1(W_F, G(\bar{L}))_{\text{bsc}}$  of *basic* 1-cocycles in  $Z^1(W_F, G(\bar{L}))$ . The set of cohomology classes of  $Z^1(W_F, G(\bar{L}))_{\text{bsc}}$  is denoted by  $\mathbf{B}(G)_{\text{bsc}}$ . The definition of  $Z^1(W_F, G(\bar{L}))_{\text{bsc}}$  implies that there are natural maps

$$Z^1(W_F, G(\bar{L}))_{\text{bsc}} \longrightarrow Z^1(F, G_{\text{ad}}), \quad \text{and} \quad \mathbf{B}(G)_{\text{bsc}} \longrightarrow H^1(F, G_{\text{ad}}).$$

An *extended pure inner twist* of  $G$  is a pair  $(\xi, z)$  where  $\xi : G \rightarrow G'$  is an inner twist and  $z$  is in  $Z^1(W_F, G(\bar{L}))_{\text{bsc}}$  such that the image of  $z$  in  $Z^1(F, G_{\text{ad}})$  is the cocycle  $\gamma \mapsto \xi^{-1}\gamma(\xi)$ . The map  $(\xi, z) \mapsto z$  establishes a bijection between the set of isomorphism classes of pure inner twists of  $G$  and the set  $\mathbf{B}(G)_{\text{bsc}}$ .

**Regular depth-zero supercuspidal  $L$ -packet data.** From now on, let  $G$  be a connected unramified reductive group over  $F$ . Let  $\varphi : W_F \rightarrow {}^L G$  be a Langlands parameter and  $\varphi_0$  its projection to  $\widehat{G}$ . Recall that  $\varphi$  is called a *TRSELP* if the restriction of  $\varphi_0$  to the wild inertia subgroup of  $W_F$  is trivial, the centralizer of  $\varphi_0(I_F)$  in  $\widehat{G}$  is a maximal torus, and the index of  $Z(\widehat{G})^\Gamma$  in  $Z_\varphi$  is finite. The notion TRSELP is the abbreviation of “tame regular semisimple elliptic Langlands parameter”, which was first introduced in [DR09, page 825].

Now let  $\varphi$  be a TRSELP. Choose a hyperspecial vertex  $o$  in  $\mathcal{B}^{\text{red}}(G, F)$ . The vertex  $o$  determines an  $O_F$ -structure on  $G$ . A *regular depth-zero supercuspidal  $L$ -packet datum* associated with  $\varphi$  is a quadruple

$$(S, {}^L j, a, \chi)$$

such that

- $S$  is an unramified elliptic maximal torus of  $G$  defined over  $O_F$ ,
- ${}^L j$  is an unramified  $L$ -embedding  ${}^L S \rightarrow {}^L G$ ,

- $a$  is a Langlands parameter  $W_F \rightarrow {}^L S$  satisfying

$${}^L j \circ a = \varphi,$$

- $\chi : S(F) \rightarrow \mathbb{C}^\times$  is the character attached to the parameter  $a$  by the local Langlands correspondence for tori.

The conditions on  $\varphi$  implies that  $\chi$  is regular. The choices of regular depth-zero supercuspidal  $L$ -packet data depend on the vertex  $o$ , and are unique up to  $G(O_F)$ -conjugate ([Kal14, Lemma 3.4.1]) after fixing a vertex.

**Regular depth-zero supercuspidal  $L$ -packets.** Let  $\varphi$  be a TRSELP, and  $(S, {}^L j, a, \chi)$  a regular depth-zero supercuspidal  $L$ -packet datum attached to  $\varphi$ . Now we review the construction of the enlarged  $L$ -packet  $\Pi_\varphi$ . For each  $b \in \mathbf{B}(G)_{\text{bsc}}$ , denote by  $G_b$  a representative in the isomorphism class of the extended pure inner twists corresponding to  $b$ . The packet  $\Pi_\varphi$  is defined to be a disjoint union of the  $L$ -packets  $\Pi_\varphi(G_b)$  where  $b$  runs over  $\mathbf{B}(G)_{\text{bsc}}$  and  $\Pi_\varphi(G_b)$  consists of certain depth-zero supercuspidal representations of  $G_b(F)$ . The construction of  $\Pi_\varphi(G_b)$  is as follows.

First let us recall how every element  $\mu$  of  $Z_\varphi^D$  gives rise to a cocycle  $z_\mu$  in  $Z^1(W_F, G(\bar{L}))_{\text{bsc}}$  (cf. [Kal14, §3.3]). The map  ${}^L j$  induces a bijection

$$Z_\varphi^D \rightarrow X_*(S)_\Gamma, \quad \mu \mapsto \bar{\lambda}_\mu. \quad (2)$$

Choose a lift  $\lambda_\mu$  of  $\bar{\lambda}_\mu$  in  $X_*(S)$  arbitrarily. The choice of  $\lambda_\mu$  does not matter (see [Kal14, Lemma 3.3.2]). The assignment  $\Phi \mapsto \lambda_\mu(\varpi)$  extends to a 1-cocycle  $W_F \rightarrow S(\bar{L})$  and its prolongation to  $G(\bar{L})$  is  $z_\mu$ , where  $\Phi \in W_F$  is the inverse of a fixed Frobenius automorphism and  $\varpi \in O_F$  is a fixed uniformizer. Let  $\xi_\mu : G \rightarrow G_\mu$  be the inner twist determined by  $z_\mu$ . Denote by

$$\mathbf{b} : Z_\varphi^D \rightarrow \mathbf{B}(G)_{\text{bsc}}$$

the map induced by  $\mu \mapsto z_\mu$ . Actually the map  $\mathbf{b}$  is the composition of the natural map  $Z_\varphi^D \rightarrow \left(Z(\hat{G})^\Gamma\right)^D$  and the isomorphism

$$\left(Z(\hat{G})^\Gamma\right)^D \xrightarrow{\sim} \mathbf{B}(G)_{\text{bsc}}.$$

Next for each  $\mu \in Z_\varphi^D$ , consider the embedding of  $S$  into  $G_\mu$

$$j_\mu : S \rightarrow G \xrightarrow{\xi_\mu} G_\mu,$$

which is in fact an admissible embedding defined over  $F$ . Denote by  $S_\mu$  the image of  $S$  in  $G_\mu$ . Then  $S_\mu$  is an elliptic unramified maximal torus of  $G_\mu$ , and isomorphic to  $S$  over  $F$ . Set  $\chi_\mu = \chi \circ j_\mu^{-1}$ , which is a character of  $S_\mu(F)$ . Based on  $(S_\mu, \chi_\mu)$ , we can construct an irreducible depth-zero supercuspidal representation  $\pi(S_\mu, \chi_\mu)$  of  $G_\mu(F)$ . The condition on  $\varphi$  implies that  $\chi$  and thus  $\chi_\mu$  are of depth zero and regular. Let  $x = \mathcal{A}^{\text{red}}(S_\mu, F)$ , which is a vertex in  $\mathcal{B}^{\text{red}}(G_\mu, F)$  since  $S_\mu$  is unramified and elliptic (cf. [DeB06, §2.2]). Let  $\mathbf{G}_\mu$  be the corresponding connected reductive group over  $k_F$  associated to  $x$ , and  $\mathbf{S}_\mu$  the maximal torus in  $\mathbf{G}_\mu$  corresponding to  $S_\mu$ . The character  $\chi_\mu$  gives rise to a



regular character  $\overline{\chi_\mu} : S_\mu(k_F) \rightarrow \mathbb{C}^\times$ , and thus an irreducible cuspidal representation  $\overline{\rho_\mu}$  of  $G_\mu(k_F)$  by Deligne-Lusztig theory. Denote by  $\rho_\mu$  the inflation of  $\overline{\rho_\mu}$  to  $G_\mu(F)_{x,0}$ . Set

$$\pi(S_\mu, \chi_\mu) = \text{ind}_{Z(F)G_\mu(F)_{x,0}}^{G_\mu(F)}(\rho_\mu \otimes \chi_\mu),$$

where  $Z$  is the center of  $G_\mu$ , and the induction is the compact induction.

Lastly, set

$$\Pi_\varphi(G_b) = \{ \pi(S_\mu, \chi_\mu) : \mu \in Z_\varphi^D \text{ such that } \mathbf{b}(\mu) = b \}, \quad (3)$$

and set

$$\Pi_\varphi = \bigsqcup_{b \in \mathbf{B}(G)_{\text{bsc}}} \Pi_\varphi(G_b). \quad (4)$$

**Enlarged local Langlands correspondence.** According to the definition of  $\Pi_\varphi$  (4), we have a bijection

$$\iota : Z_\varphi^D \longrightarrow \Pi_\varphi, \quad \mu \mapsto \pi(S_\mu, \chi_\mu),$$

which depends only on the  $G(F)$ -orbit of the chosen hyperspecial vertex  $o$ . Moreover the image  $\pi(S, \chi)$  of the trivial character under  $\iota$  is generic. More precisely, let  $(B, \psi)$  be a Whittaker datum which satisfies that the reduced apartment of the maximal torus of  $B$  contains  $o$ , and  $\psi : U(F) \rightarrow \mathbb{C}^\times$  reduces to a generic character of  $U(k_F)$ . Then  $\pi(S, \chi)$  is  $(B, \psi)$ -generic [DR09, Lemma 6.2.1]. We will denote  $\iota$  by  $\iota_{B, \psi}$  to indicate the base point. As [Kal14, Theorem 4.3.3] shows, the parametrization  $\iota_{B, \psi}$  makes the  $L$ -packet  $\Pi_\varphi$  satisfying the endoscopic character identities. In summary we have the following commutative diagram

$$\begin{array}{ccc} Z_\varphi^D & \xrightarrow{\iota_{B, \psi}} & \bigsqcup_{b \in \mathbf{B}(G)_{\text{bsc}}} \Pi_\varphi(G_b) \\ \downarrow & & \downarrow \\ (Z(\widehat{G})^\Gamma)^D & \longrightarrow & \mathbf{B}(G)_{\text{bsc}} \end{array}$$

### 2.3 $L$ -packets of the contragredient

For a general connected reductive group  $G$  over  $F$  and a Langlands parameter  $\varphi$  for  $G$ , Adams and Vogan [AV16] conjecture that the contragredient  $\Pi_\varphi^\vee(G)$  of its  $L$ -packet  $\Pi_\varphi(G)$ , defined by

$$\Pi_\varphi^\vee(G) = \{ \pi^\vee : \pi \in \Pi_\varphi(G) \},$$

is also an  $L$ -packet whose Langlands parameter is the composition of  $\varphi$  with the Chevalley involution of  ${}^L G$ . Moreover, Prasad [Pra] and Kaletha [Kal13] conjecture that the refined Langlands parameter for each representation  $\pi^\vee$  in  $\Pi_\varphi^\vee(G)$  can be explicitly determined. When  $G$  is unramified and  $\varphi$  is a TRSELP, we briefly review the enlarged local Langlands correspondence for  $\Pi_\varphi^\vee$  (defined in the same way as  $\Pi_\varphi^\vee(G)$ ), which is shown in [Kal13, §5].



**Chevalley involution.** Let  $G$  be quasi-split. To define a Chevalley involution, fix an  $F$ -splitting  $(T, B, \{X_\alpha\})$  (see [Kot84, §1.3] for the definition of splitting). The Chevalley involution  $C$  attached to this splitting is the unique involution on  $G$  defined over  $F$  such that the restriction of  $C$  to the maximal torus  $T$  is the inverse map,  $C(B) = B^{\text{op}}$  where  $B^{\text{op}}$  is the opposite of the Borel group  $B$ , and  $C(X_\alpha) = X_{-\alpha}$ . Chevalley involutions attached to different  $F$ -splittings are all  $G(\bar{F})$ -conjugate. We refer to [AV16, §2] for more information about the Chevalley involution.

Similarly, fixing an  $F$ -splitting  $(\hat{T}, \hat{B}, \{X_{\hat{\alpha}}\})$  for the complex dual group  $\hat{G}$ , we have a unique Chevalley involution  $\hat{C}$  which commutes with the action of  $\Gamma$  on  $\hat{G}$ . Thus we obtain an  $L$ -automorphism  ${}^L C = \hat{C} \times \text{id}_{W_F}$  of  ${}^L G$ . For simplicity, we will use  $C$  to denote  $\hat{C}$  or  ${}^L C$  if there is no confusion.

**Langlands parameter.** Now suppose that  $G$  is unramified and  $\varphi$  is a TRSELP for  $G$ . Let  $(S, {}^L j, a, \chi)$  be a regular depth-zero supercuspidal  $L$ -packet datum for  $\varphi$ . Choose an  $F$ -splitting  $(\hat{T}, \hat{B}, \{X_{\hat{\alpha}}\})$  for  $\hat{G}$  so that  $\hat{T} = {}^L j(\hat{S})$ , and let  $C$  be the Chevalley involution on  ${}^L G$  with respect to this splitting. Then the contragredient  $\Pi_\varphi^\vee$  of the enlarged  $L$ -packet  $\Pi_\varphi$  is the enlarged  $L$ -packet associated with the Langlands parameter  $C \circ \varphi$ . In other words, we have

$$\Pi_\varphi^\vee = \Pi_{C \circ \varphi}.$$

The reason is as follows. The parameter  $C \circ \varphi$  is a TRSELP, whose regular depth-zero supercuspidal  $L$ -packet datum can be chosen to be  $(S, {}^L j, a^{-1}, \chi^{-1})$  where  $a^{-1}$  is the Langlands parameter corresponding to  $\chi^{-1}$ . Note that  $Z_\varphi = Z_{C \circ \varphi}$ . For each  $\mu \in Z_\varphi^D$ , which is also viewed as an element in  $Z_{C \circ \varphi}^D$ , we have

$$\pi(S_\mu, \chi_\mu)^\vee = \pi(S_\mu, \chi_\mu^{-1}) = \pi(S_\mu, (\chi^{-1})_\mu). \quad (5)$$

Also note that, since  $\pi(S, \chi)$  is  $(B, \psi)$ -generic,  $\pi(S, \chi)^\vee$  is  $(B, \psi^{-1})$ -generic. Therefore, the relation between the refined local Langlands correspondence for

$$\iota_{B, \psi^{-1}} : Z_{C \circ \varphi}^D \longrightarrow \Pi_{C \circ \varphi}$$

and

$$\iota_{B, \psi} : Z_\varphi^D \longrightarrow \Pi_\varphi$$

is

$$\iota_{B, \psi^{-1}}(C \circ \varphi, \mu) = \iota_{B, \psi}(\varphi, \mu)^\vee. \quad (6)$$

## 2.4 $L$ -packets of the Galois conjugate

From now on, until the end of the paper, let  $H$  be an unramified connected reductive group over  $F$ ,  $E$  an unramified quadratic field extension of  $F$ , and  $G = R_{E/F}H$  which is also unramified over  $F$ .

**Galois conjugate.** The restriction map

$$H^1(W_F, H(\bar{L})) \longrightarrow H^1(W_E, H(\bar{L}))$$

gives rise to

$$\text{res} : \mathbf{B}(H)_{\text{bsc}} \longrightarrow \mathbf{B}(G)_{\text{bsc}}.$$

Denote by  $\mathbf{B}(G)_{\text{bsc}}^\circ$  the image of  $\mathbf{B}(H)_{\text{bsc}}$  under the restriction map. Note that  $b$  lies in  $\mathbf{B}(G)_{\text{bsc}}^\circ$  if and only if there exists  $a \in \mathbf{B}(H)_{\text{bsc}}$  such that  $G_b = R_{E/F}H_a$ , and the inner twist  $\xi_b : G \rightarrow G_b$  is induced from the inner twist  $\xi_a : H \rightarrow H_a$ . The reason that we introduce the notion  $\mathbf{B}(G)_{\text{bsc}}^\circ$  is that we want to consider the Galois action of  $\text{Gal}(E/F)$  on  $G_b$ . For  $b \in \mathbf{B}(G)_{\text{bsc}}^\circ$  and any  $a \in \mathbf{B}(H)_{\text{bsc}}$  such that  $\text{res}(a) = b$ , the  $F$ -structure of  $H_a$  induces an action, denoted by  $\sigma_a$ , of  $\sigma$  on  $G_b$ . Moreover, if there are two elements  $a$  and  $a'$  of  $\mathbf{B}(H)_{\text{bsc}}$  such that  $\text{res}(a) = \text{res}(a') = b$ , then there exists  $g \in G_b(F)$  such that  $\sigma_a = \text{Ad}(g) \circ \sigma_{a'}$ . Therefore for any representation  $\pi$  of  $G_b(F)$  we have  $\pi^{\sigma_a} \simeq \pi^{\sigma_{a'}}$ . Hence the notion  $\pi^\sigma$  is well-defined for equivalence classes of representations of  $G_b(F)$  where  $b \in \mathbf{B}(G)_{\text{bsc}}^\circ$ .

Let  $\varphi : W_F \rightarrow {}^L G$  be a TRSELP. We consider a subset  $Z_\varphi^{D,\circ}$  of  $Z_\varphi^D$ , which is defined to be

$$Z_\varphi^{D,\circ} = \{ \mu \in Z_\varphi^D : \mathbf{b}(\mu) \in \mathbf{B}(G)_{\text{bsc}}^\circ \}.$$

Set

$$\Pi_\varphi^\circ = \bigsqcup_{b \in \mathbf{B}(G)_{\text{bsc}}^\circ} \Pi_\varphi(G_b).$$

For each  $b \in \mathbf{B}(G)_{\text{bsc}}^\circ$ , define

$$\Pi_\varphi^\sigma(G_b) = \{ \pi^\sigma : \pi \in \Pi_\varphi(G_b) \}.$$

Set

$$\Pi_\varphi^{\circ,\sigma} = \bigsqcup_{b \in \mathbf{B}(G)_{\text{bsc}}^\circ} \Pi_\varphi^\sigma(G_b).$$

**Langlands parameter.** We identify  ${}^L G$  with  $(\widehat{H} \times \widehat{H}) \rtimes W_F$ . The involution  $\theta$  on  $G$  induces an  $L$ -automorphism  $\delta$  on  ${}^L G$ :

$$\delta(x, y, w) = (y, x, w), \quad \text{for } x, y \in \widehat{H}, w \in W_F.$$

Then the composition  $\delta \circ \varphi$  is also a TRSELP. Define a subset  $Z_{\delta \circ \varphi}^{D,\circ}$  of  $Z_{\delta \circ \varphi}^D$  to be

$$\{ \mu \in Z_{\delta \circ \varphi}^D : \mathbf{b}(\mu) \in \mathbf{B}(G)_{\text{bsc}}^\circ \},$$

and set

$$\Pi_{\delta \circ \varphi}^\circ = \bigsqcup_{b \in \mathbf{B}(G)_{\text{bsc}}^\circ} \Pi_{\delta \circ \varphi}(G_b).$$

The rest of this section is devoted to proving the following proposition as well as a more precise statement (Proposition 2.3), which basically shows that  $\Pi_\varphi^{\circ,\sigma}(G_b)$  for  $b \in \mathbf{B}(G)_{\text{bsc}}^\circ$  is an  $L$ -packet whose Langlands parameter is  $\delta \circ \varphi$ . The arguments are analogous to those in [Kal13, §5], which we have recalled in §2.3.

**Proposition 2.1.** *We have*

$$\Pi_\varphi^{\circ,\sigma} = \Pi_{\delta \circ \varphi}^\circ.$$

*Proof.* Recall that, to construct the bijection  $\iota_{B,\psi} : Z_\varphi^D \rightarrow \Pi_\varphi$ , we need to fix a hyperspecial vertex  $o \in \mathcal{B}^{\text{red}}(G, F)$  and a Whittaker datum  $(B, \psi)$ . Now we require that  $o$  and  $B$  (and thus  $U$ ) are  $\sigma$ -stable.

First we choose a regular depth-zero supercuspidal  $L$ -packet datum  $(S, {}^L j, a, \chi)$  for  $\varphi$ . Then  $S^\sigma$  is also an unramified elliptic maximal torus of  $G$  defined over

$O_F$ , and  $\chi^\sigma = \chi \circ \sigma$  is a character of  $S^\sigma(F)$ . Let  $a^\sigma : W_F \rightarrow {}^L(S^\sigma)$  be the Langlands parameter associated with  $\chi^\sigma$ . Denote by  $({}^L S)^\delta$  the image of  ${}^L S$  under  $\delta$  in  ${}^L G$ . According to the construction of  $L$ -groups with respect to restriction of scalars, we can identify  ${}^L(S^\sigma)$  with  $({}^L S)^\delta$ , and we have

$$a^\sigma = \delta \circ a.$$

Therefore, we have the following commutative diagram

$$\begin{array}{ccc} {}^L(S^\sigma) & \xrightarrow{{}^L j^\delta} & {}^L G \\ \delta \uparrow & & \uparrow \delta \\ {}^L S & \xrightarrow{{}^L j} & {}^L G \\ a \uparrow & & \uparrow \varphi \\ W_F & \xrightarrow{=} & W_F \end{array}$$

where

$${}^L j^\delta := \delta \circ {}^L j \circ \delta^{-1}$$

is also an unramified  $L$ -embedding. In summary, we obtain that  $(S^\sigma, {}^L j^\delta, a^\sigma, \chi^\sigma)$  is a regular depth-zero supercuspidal  $L$ -packet datum for the parameter  $\delta \circ \varphi$ .

Each character  $\mu \in Z_\varphi^D$  induces a character  $\mu \circ \delta^{-1}$  in  $Z_{\delta \circ \varphi}^D$ . Now we discuss the relation between  $(S^\sigma)_{\mu \circ \delta^{-1}}$  and  $S_\mu$ , and the relation between  $(\chi^\sigma)_{\mu \circ \delta^{-1}}$  and  $\chi_\mu$ . First we have the following commutative diagram

$$\begin{array}{ccc} Z_\varphi^D & \longrightarrow & X_*(S)_\Gamma \\ \downarrow & & \downarrow \\ Z_{\delta \circ \varphi}^D & \longrightarrow & X_*(S^\sigma)_\Gamma \end{array}$$

where the top and the bottom maps are  $\mu \mapsto \bar{\lambda}_\mu$  in (2), and the right map is  $\bar{\lambda} \mapsto \overline{\sigma \circ \lambda}$ . It is easy to check that the 1-cocycles  $z_\mu$  and  $z_{\mu \circ \delta^{-1}}$  in  $Z^1(W_F, G(\bar{L}))$ , and the inner twists  $\xi_\mu$  and  $\xi_{\mu \circ \delta^{-1}}$ , which are determined by  $\mu$  and  $\mu \circ \delta^{-1}$  respectively, satisfy the relation

$$z_{\mu \circ \delta^{-1}} = \sigma \circ z_\mu, \quad \xi_{\mu \circ \delta^{-1}} = \sigma \circ \xi_\mu \circ \sigma.$$

Thus the admissible embedding  $j_{\mu \circ \delta^{-1}}$  of  $S^\sigma$  attached to  $\mu \circ \delta^{-1}$  is

$$S^\sigma \hookrightarrow G \xrightarrow{\xi_{\mu \circ \delta^{-1}}} (G_\mu)^\sigma.$$

Hence we see that if  $\mu$  is in  $Z_\varphi^{D, \circ}$  then  $\mu \circ \delta^{-1}$  is in  $Z_{\delta \circ \varphi}^{D, \circ}$  and

$$\mathbf{b}(\mu \circ \delta^{-1}) = \mathbf{b}(\mu).$$

Moreover we have

$$(S^\sigma)_{\mu \circ \delta^{-1}} = (S_\mu)^\sigma, \quad (\chi^\sigma)_{\mu \circ \delta^{-1}} = (\chi_\mu)^\sigma.$$

Therefore,

$$\pi((S^\sigma)_{\mu \circ \delta^{-1}}, (\chi^\sigma)_{\mu \circ \delta^{-1}}) = \pi((S_\mu)^\sigma, (\chi_\mu)^\sigma) \quad (7)$$

From now on, we simply denote

$$S_\mu^\sigma = (S_\mu)^\sigma, \quad \chi_\mu^\sigma = (\chi_\mu)^\sigma.$$

In summary, according to (7), we have shown that

$$\Pi_{\delta \circ \varphi}^\circ = \{\pi(S_\mu^\sigma, \chi_\mu^\sigma) : \mu \in Z_\varphi^{D, \circ}\}.$$

Hence Proposition 2.1 follows from the lemma below directly.  $\square$

**Lemma 2.2.** *For any  $\mu \in Z_\varphi^{D, \circ}$ , we have*

$$\pi(S_\mu^\sigma, \chi_\mu^\sigma) \simeq \pi(S_\mu, \chi_\mu)^\sigma.$$

*Proof.* For any open subgroup  $K$  of  $G_b(F)$  which is compact modulo the center, and any representation  $(\rho, V_\rho)$  of  $K$ , we have

$$\left(\text{ind}_K^{G_b(F)} \rho\right)^\sigma \simeq \text{ind}_{K^\sigma}^{G_b(F)} (\rho^\sigma), \quad (8)$$

where  $K^\sigma$  is the image of  $K$  under  $\sigma$ , and  $(\rho^\sigma, V_{\rho^\sigma}) = (\rho \circ \sigma, V_\rho)$  is the representation of  $K^\sigma$ . The intertwining operator is given by

$$i : f \mapsto f \circ \sigma,$$

where  $f : G_b(F) \rightarrow V_\rho$  is any function in the space of  $\left(\text{ind}_K^{G_b(F)} \rho\right)^\sigma$ . Let  $G_b(F)_{x,0}$  be the parahoric subgroup corresponding to  $S_\mu$ . Then we have

$$G_b(F)_{x,0}^\sigma = G_b(F)_{\sigma(x),0},$$

where  $G_b(F)_{\sigma(x),0}$  is the parahoric subgroup corresponding to  $S_\mu^\sigma$ . Let  $\rho_\mu$  be the representation of  $G_b(F)_{x,0}$  constructed via  $(S_\mu, \chi_\mu)$ , and  $\rho_{\mu,\sigma}$  the representation of  $G_b(F)_{\sigma(x),0}$  constructed via  $(S_\mu^\sigma, \chi_\mu^\sigma)$ . It is straightforward that  $\rho_{\mu,\sigma} \simeq \rho_\mu^\sigma$ , which implies the lemma by (8).  $\square$

Since  $\pi(S, \chi)$  is  $(B, \psi)$ -generic, the representation  $\pi(S^\sigma, \chi^\sigma)$  is  $(B, \psi^\sigma)$ -generic. Hence it makes sense to denote both of the two bijective maps

$$Z_{\delta \circ \varphi}^D \longrightarrow \Pi_{\delta \circ \varphi} \quad \text{and} \quad Z_{\delta \circ \varphi}^{D, \circ} \longrightarrow \Pi_{\delta \circ \varphi}^\circ$$

by  $\iota_{B, \psi^\sigma}$ . The following proposition, which is finer than Proposition 2.1, is a direct consequence of (7) and Lemma 2.2.

**Proposition 2.3.** *For any  $\mu \in Z_\varphi^{D, \circ}$ , we have*

$$\iota_{B, \psi}(\varphi, \mu)^\sigma = \iota_{B, \psi^\sigma}(\delta \circ \varphi, \mu \circ \delta^{-1}).$$

### 3 Distinction

#### 3.1 Finite fields

In this subsection, we recall a result on the problem of distinction of representations for groups over finite fields. Let  $\mathbf{H}$  be a connected reductive group over a finite field  $k$ , and  $k'$  a quadratic field extension of  $k$ . Denote  $\mathbf{G} = \mathbf{R}_{k'/k}\mathbf{H}$ . Then the Frobenius map  $F$  corresponding to the  $k$ -structure of  $\mathbf{H}$  defines an involution  $\theta$  on  $\mathbf{G}$ . Let  $\mathbf{S}$  be an elliptic maximal torus of  $\mathbf{G}$  over  $k$ , and  $\chi : \mathbf{S}(k) \rightarrow \mathbb{C}^\times$  a regular character. Then, by Deligne-Lusztig theory, there is an irreducible cuspidal representation  $\rho$  of  $\mathbf{G}(k)$  associated to  $(\mathbf{S}, \chi)$ . The following lemma is a special case of [Lus00, Lemma 2.2].

**Lemma 3.1.** *The space  $\text{Hom}_{\mathbf{H}(k)}(\rho, 1)$  is nonzero if and only if there exists  $g \in \mathbf{G}(k)$  such that  $\mathbf{S}^g = g\mathbf{S}g^{-1}$  is an  $F$ -stable torus, and*

$$\chi^g|_{\mathbf{S}^g(k)^\theta} = 1$$

where  $\chi^g = \chi \circ \text{Ad}(g^{-1})$ .

The main theorem of [Lus00] assumes that the center  $Z$  of  $\mathbf{G}$  is connected and  $\mathbf{G}/Z$  is simple. For [Lus00, Lemma 2.2], this assumption is not needed.

#### 3.2 Proof of Theorem 1.2

Let  $\varphi : W_F \rightarrow {}^L G$  be a TRSELP, and  $\Pi_\varphi$  the corresponding enlarged  $L$ -packet. Fix a character  $\mu \in Z_\varphi^{D,\circ}$ , and let  $b = \mathbf{b}(\mu)$ . Choose  $a \in \mathbf{B}(H)_{\text{bsc}}$  such that  $\text{res}(a) = b$ . Only in this subsection, by abuse of notation, we denote

$$(\pi, S, \chi, G, H) = (\pi(S_\mu, \chi_\mu), S_\mu, \chi_\mu, G_b, H_a)$$

for simplicity. The following proposition, which is analogous to Lemma 3.1, is the key point for the proof of Theorem 1.2.

**Proposition 3.2.** *Suppose that  $\pi$  is  $H(F)$ -distinguished. Then there exists  $g \in G(F)$  such that  $\mathbf{S}^g$  is  $\theta$ -stable and*

$$\chi^g|_{\mathbf{S}^g(F)^\theta} = 1.$$

*Proof.* Let  $x = \mathcal{A}^{\text{red}}(S, F)$ , which is a vertex in  $\mathcal{B}^{\text{red}}(G, F)$ . Write  $K$  for  $Z(F)G(F)_{x,0}$ , and  $\tilde{K}$  for  $G(F)_x$  which is the stabilizer of  $x$  for the action of  $G(F)$  on  $\mathcal{B}^{\text{red}}(G, F)$ . It is known that  $\tilde{K}$  is the normalizer of  $K$  in  $G(F)$  and  $K$  is of finite index in  $\tilde{K}$ .

Since  $\pi$  is  $H(F)$ -distinguished, applying Hakim-Murnaghan theory (cf. [HM08, Theorem 5.26]), we can and do assume that  $\theta(x) = x$ , which implies that  $\tilde{K}$ ,  $K$  and  $G(F)_{x,0}$  are all  $\theta$ -stable. Therefore the vertex  $x$  lies in  $\mathcal{B}^{\text{red}}(H, F)$ . Denote by  $\mathcal{H}_x$  the canonical connected smooth group scheme over  $O_F$ , which is an integral model of  $H$ , such that  $\mathcal{H}_x(O_F) = H(F)_{x,0}$ . Since  $E$  is unramified over  $F$ , we have  $\mathcal{H}_x(O_E) = G(F)_{x,0}$ . Let  $\mathbf{G}$  and  $\mathbf{H}$  be the corresponding connected reductive groups over  $k_F$ . Then  $\mathbf{G} = \mathbf{R}_{k_E/k_F}\mathbf{H}$ . The involution  $\theta$  acts on  $G(F)_{x,0}$ , and induces an involution  $\bar{\theta}$  on  $\mathbf{G}$ . Moreover the involution  $\bar{\theta}$  is the Frobenius map  $F$  with respect to the  $k_F$ -structure of  $\mathbf{H}$ .

By abuse of notation, we use  $\rho$  to denote both of the representations  $\rho_\mu \otimes \chi_\mu$  of  $K$  and  $\rho_\mu$  of  $G(F)_{x,0}$ . Let  $\tilde{\rho} = \text{ind}_K^{\tilde{K}} \rho$ , which is an irreducible representation of  $\tilde{K}$ . According to [HM08, Theorem 5.26], we have

$$\text{Hom}_{H(F)}(\pi, 1) \simeq \bigoplus_{g_i} \text{Hom}_{\tilde{K} \cap G(F)^{\theta_i}}(\tilde{\rho}, 1) = \bigoplus_{g_i} \text{Hom}_{\tilde{K}^{\theta_i}}(\tilde{\rho}, 1),$$

where  $g_i$  runs over representatives of the double cosets  $\tilde{K} \backslash G(F) / H(F)$  such that  $\tau(g_i) \in \tilde{K}$  (which does not depend on the choice of the representatives), and  $\theta_i$ 's are the  $g_i$ -twisted involutions defined by  $\theta_i(y) = \tau(g_i)^{-1} \theta(y) \tau(g_i)$ . Recall that  $\tau(g)$  is defined to be  $g\theta(g)^{-1}$ . Applying Mackey theory, we have a finer decomposition

$$\text{Hom}_{H(F)}(\pi, 1) \simeq \bigoplus_{g_i} \text{Hom}_{K^{\theta_i}}(\rho, 1)$$

where each  $g_i \in K \backslash G(F) / H(F)$  satisfies  $\tau(g_i) \in \tilde{K}$ .

For each  $i$ , write  $x_i = \tau(g_i)$ . Then  $\theta_i = \text{Ad}(x_i^{-1}) \circ \theta$ . Denote by  $\bar{\theta}_i$  the involution on  $\mathbf{G}(k_F) = \mathbf{H}(k_E)$  induced by  $\theta_i$ . Now we want to show that  $\bar{\theta}_i$  is actually a Frobenius map on  $\mathbf{H}(\bar{k}_F)$ . For any  $\bar{y} \in \mathbf{H}(\bar{k}_F)$  let  $y \in \mathcal{H}_x(O_{F^u})$  be a lift of  $\bar{y}$ . We view  $\theta$  as the automorphism on  $\mathcal{H}_x(F^u)$  induced by the Frobenius map in  $\text{Gal}(F^u/F)$ . Thus we have

$$\bar{\theta}_i^2(\bar{y}) = \overline{x_i^{-1} \theta(x_i^{-1} \theta(y) x_i)} x_i = \overline{\theta^2(y)} = F^2(\bar{y}).$$

Therefore, for each  $i$ ,  $\bar{\theta}_i$  is a Frobenius map. Denote  $\bar{\theta}_i$  by  $F_i$  for simplicity. Then  $\mathbf{G}(k_F) = \mathbf{H}(\bar{k}_F)^{F_i}$ . Let  $\mathbf{H}_i$  be the connected reductive group over  $k_F$  such that  $\mathbf{H}_i(k_F) = \mathbf{H}(\bar{k}_F)^{F_i}$ .

For each  $i$ , if  $\text{Hom}_{K^{\theta_i}}(\rho, 1)$  is nonzero then  $\text{Hom}_{G(F)_{x,0}^{\theta_i}}(\rho, 1)$  is nonzero. Note that

$$\text{Hom}_{G(F)_{x,0}^{\theta_i}}(\rho, 1) = \text{Hom}_{\mathbf{H}_i(k_F)}(\bar{\rho}, 1).$$

Since  $\text{Hom}_{H(F)}(\pi, 1)$  is nonzero, there exists an index  $i$  such that  $\text{Hom}_{\mathbf{H}_i(k_F)}(\bar{\rho}, 1)$  is nonzero. As before, let  $\mathbf{S}$  be the elliptic maximal torus in  $\mathbf{G}$  corresponding to  $S$ , and  $\bar{\chi}$  the character of  $\mathbf{S}(k_F)$  induced by  $\chi$ . According to Lemma 3.1, there exists an  $F_i$ -stable torus  $\mathbf{S}'$  in  $\mathbf{G}$  and a character  $\bar{\chi}'$  of  $\mathbf{S}'(k_F)$  satisfying

$$\bar{\chi}'|_{\mathbf{S}'(k_F)^{F_i}} = 1$$

such that  $(\mathbf{S}, \bar{\chi})$  and  $(\mathbf{S}', \bar{\chi}')$  are  $\mathbf{G}(k_F)$ -conjugate.

By [HL12, Lemma A.2], there exists a  $\theta_i$ -stable elliptic unramified maximal torus  $S'$  of  $G$  such that  $x = \mathcal{A}^{\text{red}}(S', F)$  and the image of  $S'(F^u) \cap G(F^u)_{x,0}$  in  $\mathbf{G}(\bar{k}_F)$  is  $\mathbf{S}'(\bar{k}_F)$ . Since  $\mathbf{S}$  and  $\mathbf{S}'$  are  $\mathbf{G}(k_F)$ -conjugate,  $S$  and  $S'$  are  $G(F)_{x,0}$ -conjugate. Choose  $g_0 \in G(F)_{x,0}$  such that  $S' = S^{g_0}$ . Let  $\chi' = \chi^{g_0}$ . It is easy to check that  $\bar{\chi}' = \bar{\chi}$ . Therefore  $\bar{\chi}'|_{\mathbf{S}'(k_F)^{F_i}} = 1$  and thus

$$\chi'|_{S'(O_F)^{\theta_i}} = 1.$$

Let  $Z'$  be the center of  $H$ . The central character of  $\pi$  is  $\chi|_{Z(F)}$ , which is equal to  $\chi'|_{Z(F)}$ . Since  $\pi$  is  $H(F)$ -distinguished, we have

$$\chi'|_{Z'(F)} = 1.$$

Since  $S'$  is elliptic and unramified, we have  $S'(F)^{\theta_i} = Z'(F)S'(O_F)^{\theta_i}$ . Therefore

$$\chi'|_{S'(F)^{\theta_i}} = 1.$$

Recall that  $\theta_i = \text{Ad}(x_i^{-1}) \circ \theta$  with  $x_i = \tau(g_i)$ . Let  $g' = \theta(g_i)^{-1}$ . It is obvious that  $S'' = S'^{g'}$  is  $\theta$ -stable and  $S''(F)^{\theta} = g'S'(F)^{\theta_i}g'^{-1}$ . Let  $\chi'' = \chi'^{g'}$ . Then

$$\chi''|_{S''(F)^{\theta}} = 1.$$

In summary, if we set  $g = g'g_0$  then  $g$  satisfies the desired conditions in Proposition 3.2.  $\square$

**Proposition 3.3.** *If  $\pi$  is  $H(F)$ -distinguished, we have*

$$\pi^{\vee} \simeq \pi^{\sigma}.$$

*Proof.* According to [Kal14, Lemma 3.1.1], the equivalence class of  $\pi$  depends only on the  $G(F)$ -conjugate class of  $(S, \chi)$ . By Proposition 3.2, we can and do assume that  $S$  is  $\theta$ -stable and  $\chi|_{T(F)} = 1$ , where  $T = S^{\theta}$  is a maximal torus of  $H$ . The condition  $\chi|_{T(F)} = 1$  implies that  $\chi^{\sigma} = \chi^{-1}$ . Combining (5) and Lemma 2.2, we obtain

$$\pi(S, \chi)^{\sigma} \simeq \pi(S^{\sigma}, \chi^{\sigma}) = \pi(S, \chi^{-1}) \simeq \pi(S, \chi)^{\vee}.$$

Therefore,  $\pi^{\sigma} \simeq \pi^{\vee}$ .  $\square$

Now we have proved Theorem 1.2 when  $\omega_H$  is trivial. We will construct an inflation  $\omega_G$  of  $\omega_H$ , which is a character of  $G(F)$ , and reduce Theorem 1.2 to Proposition 3.3.

First let us recall the construction of  $\omega_H$ . Let  $\alpha$  be the element in  $H^1(W_F, \mathbb{Z}/2\mathbb{Z})$  associated with the quadratic extension  $E$  of  $F$ . By choosing a regular unipotent element in  $\widehat{H}_{\text{ad}}$ , we obtain a morphism  $\text{SL}_2(\mathbb{C}) \rightarrow \widehat{H}_{\text{ad}}$  and thus a morphism  $\beta : \mathbb{Z}/2\mathbb{Z} \rightarrow Z(\widehat{H}_{\text{ad}})$  by restriction to the center. It is explained in [Pra01, §7] that all characters of  $H(F)$  come from  $H^1(W_F, Z(\widehat{H}_{\text{ad}}))$  and the character  $\omega_H$  is the one attached to  $\beta \circ \alpha$ .

Let  $E'$  be the unique unramified extension of  $E$  in  $\bar{F}$  and  $\alpha'$  the element in  $H^1(W_{E'}, \mathbb{Z}/2\mathbb{Z})$  associated with  $E'$ . Similarly as the construction of  $\omega_H$ , we get a character  $\omega_G$  of  $G(F) = H(E)$  associated to  $\beta \circ \alpha'$ . Note that  $\alpha$  factors through  $W_F/I_F \simeq \mathbb{Z}$  with kernel  $W_E/I_F$ , and  $\alpha'$  factors through  $W_{E'}/I_F \simeq 2\mathbb{Z}$ , as a subgroup of  $W_F/I_F$ , with kernel  $W_{E'}/I_F$ . Hence we have the following commutative diagram

$$\begin{array}{ccc} W_F/I_F & \xrightarrow{\alpha} & \mathbb{Z}/2\mathbb{Z} \\ \downarrow & & \downarrow \\ W_{E'}/I_F & \xrightarrow{\alpha'} & \mathbb{Z}/2\mathbb{Z} \end{array}$$

where the left map is  $t \mapsto 2t$ . Therefore  $\omega_G$  is an inflation of  $\omega_H$ . Let  $\omega_G^{\sigma}$  be the Galois conjugate of  $\omega_G$  for  $\sigma \in \text{Gal}(E/F)$ . Note that  $\text{Gal}(E/F) \simeq W_F/W_E$  has a natural action on  $W_E/I_F$  and induces an action on  $H^1(W_E/I_F, \mathbb{Z}/2\mathbb{Z})$ . Then  $\omega_G^{\sigma}$  is the character of  $G(F)$  associated with  $\beta \circ \sigma(\alpha')$ . On the other hand, since  $W_F/I_F$  is abelian,  $\sigma(\alpha') = \alpha'$ . In summary, we have the following lemma.



**Lemma 3.4.** *The quadratic character  $\omega_G$  is an inflation of  $\omega_H$  and satisfies  $\omega_G^\sigma = \omega_G$ .*

*Proof of Theorem 1.2.* Suppose that  $\pi$  is  $\omega_H$ -distinguished. Since  $\omega_G$  is an inflation of  $\omega_H$ , we have

$$\mathrm{Hom}_{H(F)}(\pi \otimes \omega_G, 1) = \mathrm{Hom}_{H(F)}(\pi, \omega_G) = \mathrm{Hom}_{H(F)}(\pi, \omega_H) \neq 0.$$

In other words, we have  $\pi \otimes \omega_G$  is  $H(F)$ -distinguished. Since  $\pi \otimes \omega_G$  is also of depth zero, according to Proposition 3.3, we have

$$\pi^\vee \otimes \omega_G^{-1} \simeq (\pi \otimes \omega_G)^\vee \simeq (\pi \otimes \omega_G)^\sigma \simeq \pi^\sigma \otimes \omega_G^\sigma.$$

By Lemma 3.4, we have

$$\pi^\vee \simeq \pi^\sigma \otimes \omega_G^\sigma \omega_G = \pi^\sigma.$$

□

### 3.3 Consequences

Let  $\varphi$  be a TRSELP for  $G$  as before. Based on the description of  $\Pi_\varphi^\vee$  and  $\Pi_\varphi^{\circ, \sigma}$ , we can investigate the problem of distinction in terms of  $L$ -parameters. Recall that the hyperspecial vertex  $o$  and the Whittaker datum  $(B, \psi)$  are chosen to be  $\sigma$ -stable. Let  $B'$  be the Borel subgroup of  $H$  such that  $B = R_{E/F} B'$ , and  $U'$  the unipotent radical of  $B'$ . Then  $U = R_{E/F} U'$ . Moreover we choose  $\psi$  in the way that its restriction to  $U'(F)$  is trivial. Hence  $\psi^\sigma = \psi^{-1}$ . From now on, for  $\mu \in Z_\varphi^{D, \circ}$ , we call  $\pi(S_\mu, \chi_\mu)$  *distinguished* for short if it is  $\omega_{H_a}$ -distinguished for some  $a \in \mathbf{B}(H)_{\mathrm{bsc}}$  such that  $\mathrm{res}(a) = \mathbf{b}(\mu)$ . If  $\varphi'$  is another Langlands parameter for  $G$ , we write  $\varphi \sim \varphi'$  if these two parameters are  $\widehat{G}$ -conjugate. For  $\mu \in Z_\varphi^D$  and  $\mu' \in Z_{\varphi'}^D$ , we write  $(\varphi, \mu) \sim (\varphi', \mu')$  if there exists  $\widehat{g} \in \widehat{G}$  such that  $\varphi' = \mathrm{Ad}(\widehat{g}) \circ \varphi$  and  $\mu' = \mathrm{Ad}(\widehat{g}) \circ \mu$ .

**Theorem 3.5.** *Suppose that  $\pi(S_\mu, \chi_\mu)$  is distinguished. Then we have*

$$(C \circ \varphi, \mu) \sim (\delta \circ \varphi, \mu \circ \delta^{-1}).$$

*Proof.* First, by (6), we have

$$\pi(S_\mu, \chi_\mu)^\vee = \iota_{B, \psi^{-1}}(C \circ \varphi, \mu).$$

On the other hand, by Proposition 2.3, we have

$$\pi(S_\mu, \chi_\mu)^\sigma = \iota_{B, \psi^\sigma}(\delta \circ \varphi, \mu \circ \delta^{-1}) = \iota_{B, \psi^{-1}}(\delta \circ \varphi, \mu \circ \delta^{-1}).$$

Then the conclusion follows from Theorem 1.2. □

**The group  $H^{\mathrm{op}}$ .** Now we interpret Theorem 3.5 in the language of functoriality. By twisting the Galois structure  $\mathrm{Gal}(\bar{F}/F) \rightarrow \mathrm{Out}(H(\bar{F}))$  of  $H$  via sending  $\sigma \in \mathrm{Gal}(E/F)$  to a fixed Chevalley involution  $C$  (over  $F$ ) in  $\mathrm{Out}(H(\bar{F}))$ , we obtain a quasi-split reductive group  $H^{\mathrm{op}}$  over  $F$ , which is isomorphic to  $H$  over  $E$ . We can identify the  $L$ -group  ${}^L H^{\mathrm{op}}$  of  $H^{\mathrm{op}}$  as the subgroup

$$\left\{ (h, C(h)) : h \in \widehat{H} \right\} \rtimes W_F$$

of  ${}^L G = \left( \widehat{H} \times \widehat{H} \right) \rtimes W_F$  up to  $\widehat{G}$ -conjugacy, where  $C$  is a Chevalley involution on  $\widehat{H}$ .

The following corollary, which is a direct consequence of Theorem 3.5, shows that a regular depth-zero supercuspidal  $L$ -packet which contains a distinguished representation should be a functorial lift from  $H^{\text{op}}$  via the base change map  ${}^L H^{\text{op}} \rightarrow {}^L G$ .

**Corollary 3.6.** *If  $\pi(S_\mu, \chi_\mu)$  is distinguished for some  $\mu \in Z_\varphi^{D, \circ}$ , we have*

1.  $\Pi_{C \circ \varphi} = \Pi_{\delta \circ \varphi}$ ,
2. *The parameter  $\varphi$  factors through  ${}^L H^{\text{op}}$ .*

*Proof.* The first assertion is due to Theorem 3.5. For the second assertion, write the projection  $\varphi_0$  of  $\varphi$  to  $\widehat{G}$  as  $(\varphi_1, \varphi_2)$  so that

$$\varphi_0(w) = (\varphi_1(w), \varphi_2(w))$$

for  $w \in W_F$ . Then the condition  $C \circ \varphi \sim \delta \circ \varphi$  implies that  $\varphi_2 \sim C \circ \varphi_1$ , that is,  $\varphi$  factors through  ${}^L H^{\text{op}}$ . □

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